Implications of an inverse branching aftershock sequence model

D. L. Turcotte* and S. G. Abaimov
Department of Geology, University of California, Davis, California 95616, USA

I. Dobson†
Electrical and Computer Engineering Department, University of Wisconsin, Madison Wisconsin 53706, USA

J. B. Rundle‡
Departments of Geology and Physics, University of California, Davis, California 95616, USA
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The branching aftershock sequence (BASS) model is a self-similar statistical model for earthquake aftershock sequences. A prescribed parent earthquake generates a first generation of daughter aftershocks. The magnitudes and times of occurrence of the daughters are obtained from statistical distributions. The first generation daughter aftershocks then become parent earthquakes that generate second generation aftershocks. The process is then extended to higher generations. The key parameter in the BASS model is the magnitude difference Δm^* between the parent earthquake and the largest expected daughter earthquake. In the application of the BASS model to aftershocks Δm^* is positive, the largest expected daughter event is smaller than the parent, and the sequence of events (aftershocks) usually dies out, but an exponential growth in the number of events with time is also possible. In this paper we explore this behavior of the BASS model as Δm^* varies, including when Δm^* is negative and the largest expected daughter event is larger than the parent. The applications of this self-similar branching process to biology and other fields are discussed.

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I. INTRODUCTION

The purpose of this paper is to study the behavior of a self-similar branching process. The motivation comes from studies of aftershock sequences following earthquakes. A large number of papers have studied the behavior of the epidemic type aftershock sequence (ETAS) models [1]. This model includes multigenerational branching processes.

Branching processes have a long history and an extensive literature. Much of this work has been associated with population growth. On average the population N_i in the n_i generation is given by the exponential relation

$$N_i = N_0 R^{n_i}, \tag{1}$$

where N_0 is the initial population, $n_i = 1, 2, 3, \ldots$, and R is the basic reproductive rate (mean number of next-generation daughters per parent). For the subcritical case $R < 1$ the population always eventually dies out, and for the supercritical case $R > 1$ there is either exponential growth or the population dies out. The general case of integer numbers of daughters is known as a Galton-Watson process and this is a standard problem in statistics [2]. The exponential relation (1) determines the average number of daughters in each generation, where the average is taken over all realizations of the branching process. A realization in which the population dies out occurs when random variation yields a generation with zero daughters. There can be realizations in which the population dies out even in cases in which $R > 1$ and the average number of daughters increase. However, die out is less likely when the population is large.

The concept of branching described above has also been applied to the foreshock-mainshock-aftershock sequences of earthquakes [1,3,4]. All earthquakes produce aftershocks that, for large earthquakes, can continue for a year or longer. It is generally accepted that the aftershocks are caused by stress transfer during the main rupture. The time delays associated with aftershocks can be explained by damage mechanics. Aftershocks are found to obey three scaling laws to a good approximation [5].

(1) Gutenberg-Richter (GR) frequency-magnitude scaling [6]. This scaling is a power-law (fractal) relation between rupture area and the number of events with fractal dimension $D \approx 2$ [7]. The background seismicity that drives aftershock activity also satisfies this scaling with $D \approx 2$.

(2) Omori's law for the temporal decay of aftershock activity [8,9]. The rate of aftershock activity is approximately proportional to t^{-1} where t is the time since the mainshock occurred.

(3) Bath's law that the magnitude of the largest aftershock is approximately 1.2 magnitude units less than the mainshock [10]. Acceptance of Bath's law leads to the conclusion that aftershock sequences are self similar independent of the magnitude of the mainshock.

Foreshocks also occur systematically. The magnitude distributions of foreshocks satisfy a modified form of GR statistics and an inverse Omori's law for the time delays between foreshocks and main shocks [11]. When one or more foreshocks occur, the main shock is considered to be an aftershock that is larger than the initial triggering earthquake.

A widely accepted branching model for earthquake aftershocks is the ETAS model [1]. This model recognizes that each earthquake has an associated aftershock sequence. The

*turcotte@geology.ucdavis.edu
†dobson@engr.wisc.edu
‡rundle@cse.ucdavis.edu

original earthquake in the sequence is the parent which generates a family of daughters, the aftershocks. Each of these daughter aftershocks becomes a parent that can generate a second generation of daughter aftershocks. Higher order families of aftershocks are also generated. The sequence must die out since the energy available for the aftershocks is limited.

In the ETAS model the number of daughter aftershocks is determined from a productivity relation. A random number generator determines the magnitude of each aftershock from a GR distribution and the time of occurrence from Omori's law. The key parameter in the ETAS model is the mean number $\bar{N}$ of direct aftershocks per earthquake, averaged over all magnitudes. If $\bar{N}<1$, the sequence is subcritical and the branching process dies out; if $\bar{N}>1$, the sequence is supercritical and the branching process can explode.

The self-similar limit of the ETAS model is the branching aftershock sequence (BASS) model [4]. In this model the productivity law is replaced by a modified version of Bath's law [12]. The BASS model is fully self similar. It is the purpose of this paper to examine the behavior of the BASS model and particularly whether the aftershocks die out or increase exponentially with time.

The key parameter in the BASS model is the magnitude difference Δm^* between the parent earthquake and the largest expected aftershock. If Δm^* is negative, i.e., the largest expected aftershock is larger than the mainshock, then the number of events (aftershocks) grows exponentially with time. If Δm^* is positive and large, then the largest expected aftershock is much smaller than the mainshock or nonexistent and the sequence of aftershocks dies out. In an intermediate range of Δm^*, the sequence of aftershocks can either grow or die out. The sequence is always supercritical and the average number of events grows exponentially according to Eq. (1). Despite this behavior of the average, there is a finite and sometimes likely possibility of the events dying out. We will examine this behavior as Δm^* varies.

II. SELF-SIMILAR BRANCHING MODEL

Although our self-similar branching model is motivated by the BASS model for aftershocks, we believe that it is a more general model for branching sequences. For convenience we retain the magnitude nomenclature utilized for earthquakes. For earthquakes the event magnitude m is related to the radiated energy E in Joule by the empirical relation [13]

$$\log_{10} E = 1.5m + 4.8. \tag{2}$$

For our general event analysis we consider the magnitude to be a measure of the energy of an event. We consider an original parent event with magnitude m_p. This event will generate subsequent daughter events with magnitudes m_d.

The magnitude distribution of daughter events is assumed to satisfy a modified version of GR scaling of the form [4,12]

$$N_d(\geqslant m_d) = \lfloor 10^{b(m_p-\Delta m^*-m_d)} \rfloor, \tag{3}$$

where m_p is the magnitude of the parent event, m_d is the magnitude of the daughter event, and $N_d(\geqslant m_d)$ is the number of daughter events with magnitudes greater than or equal to m_d. The symbol $\lfloor x \rfloor$ denotes the integer part of x and its use in Eq. (3) ensures that the number of daughter events is an integer. For example, $\lfloor 4.7 \rfloor = 4$ and $\lfloor 0.2 \rfloor = 0$. The parameter Δm^* controls the number of daughter events and the proportion of cases in which $N_d(\geqslant m_d)$ becomes zero and the sequence of events ends. With $m_d = m_p - \Delta m^*$ we have $N_d(\geqslant m_d) = 1$.

For aftershock sequences the parameter b (the b-value) has a value near unity $b \approx 1$. It has also been recognized that for aftershock sequences the magnitude difference Δm^* has a near constant value $\Delta m^* \approx 1.2$. This scaling for aftershocks is known as the modified form of Bath's law [12]. It is the purpose of this paper to study the dependence of the resulting self-similar branching sequences on the value of Δm^*.

In order to constrain the sequence of aftershocks (daughter earthquakes) it is necessary to specify a minimum magnitude earthquake $m_{\min}$ in the sequence. From Eq. (3), the total number of daughter earthquakes N_{dt} is given by

$$N_{dt} = N(\geqslant m_{\min}) = \lfloor 10^{b(m_p-\Delta m^*-m_{\min})} \rfloor. \tag{4}$$

In the limit $m_{\min} \to -\infty$ the number of daughter earthquakes is infinite. From Eqs. (3) and (4) the cumulative distribution function P_m for the magnitudes of the daughter earthquakes is given by

$$P_m = \frac{N_d(\geqslant m_d)}{N_{dt}} \approx 10^{-b(m_d-m_{\min})}. \tag{5}$$

The magnitude of each of the N_{dt} daughter earthquakes is determined from this distribution.

We next determine the time of occurrence of each daughter earthquake. We require that the time delay t satisfies the generalized form of Omori's law [8]

$$R(t) = \frac{dN_d}{dt} = \frac{1}{\tau\left(1+\frac{t}{c}\right)^p}, \tag{6}$$

where $R(t)$ is the rate of aftershock occurrence and τ, c, and p are parameters. The total number of daughter aftershocks that occur after a time t is then given by

$$N_d(\geqslant m_d) = \int_t^\infty \frac{dN_d}{dt} dt = \frac{c}{\tau(p-1)\left(1+\frac{t}{c}\right)^{p-1}}. \tag{7}$$

The total number of daughter earthquakes N_{dt} is obtained by setting $t=0$ in Eq. (7) with the result

$$N_{dt} = \frac{c}{\tau(p-1)}. \tag{8}$$

From Eqs. (7) and (8) the cumulative distribution function P_t for the times of occurrence of the daughter earthquakes is given by

$$P_t = \frac{N_d(\geq t)}{N_{dt}} = \frac{1}{\left(1+\frac{t}{c}\right)^{p-1}}. \quad (9)$$

The time of occurrence after the parent earthquake of each of the N_{dt} daughter earthquakes is determined randomly from this distribution.

There are two characteristic times c and τ in the generalized form of Omori's law given in Eq. (6). The characteristic time τ specifies the initial rate of event activity at $t=0$ and the characteristic time c is a measure of the time delay before the onset of event activity. Observations for earthquakes [12] indicate that it is a good approximation to take $\tau=\tau_0$ constant for an aftershock sequence. In this case c is a function of the difference in magnitude m_p-m_d. To obtain this dependence we write Eq. (8) in the form

$$N_d(\geq m_d) = \frac{c(m_p,m_d)}{\tau_0(p-1)}. \quad (10)$$

Combining Eqs. (3) and (10) gives

$$c(m_p,m_d) = \tau_0(p-1)10^{b(m_p-m_d-\Delta m^*)}. \quad (11)$$

The time delay c increases as the magnitude difference between parent and daughter events m_p-m_d increases. There is a cascade of events from large to small with increasing time delays. Values of m_p and m_d are used to determine the values of c that is substituted into Eq. (9) to determine the time of occurrence of the daughter event.

III. BASS SIMULATIONS

Model sequences of events are generated using simulations. In order to carry out these simulations we do the following.

(1) Specify the magnitude m_p of the initial parent event. This event is introduced at $t=0$.

(2) Specify the minimum magnitude of events $m_{\min}$ to be considered.

(3) Specify the magnitude difference Δm^*. This parameter primarily controls the probability that the events die out.

(4) The total number of daughter earthquakes N_{dt} is determined using Eq. (4). These are the first generation aftershocks.

(5) For each of the N_{dt} daughter earthquakes, two random numbers in the range 0 to 1 are selected. The first random number is a value of the cumulative distribution function P_m and Eq. (5) is used to obtain the corresponding magnitude m_d. We take $b=1$ in our simulations. The characteristic delay time c is then obtained from Eq. (11). We take $p=1.25$ and set $\tau_0=1$. Our resulting times are $T=t/\tau_0$. The second random number is a value of the cumulative distribution function P_t and Eq. (9) is used to obtain the time of occurrence $t(T)$ of the daughter event. The procedure is repeated for each daughter event.

(6) Each of the first generation event is then taken to be a parent event and families of second generation events are generated using the procedure described above.

(7) The process is repeated for third generation and higher generation aftershocks.

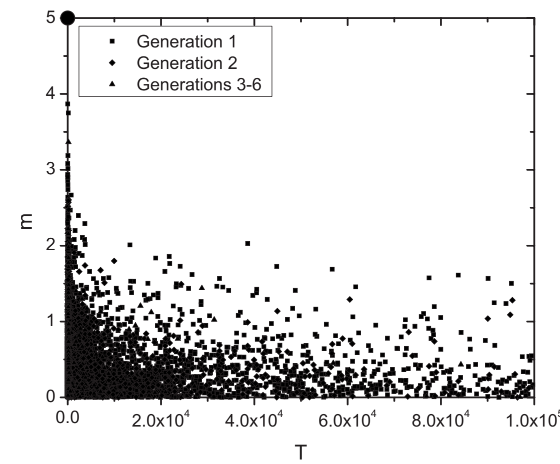

FIG. 1. Simulation of an aftershock sequence for a magnitude $m_p=5$ mainshock. Magnitudes m of the aftershocks are given as a function of the nondimensional time $T=t/\tau_0$. Six generations of aftershocks were generated.

IV. SIMULATION RESULTS

We will first generate a stable sequence of events that corresponds to a sequence of aftershocks. For this purpose we take the magnitude m_p of the original mainshock event to be $m_p=5$ and take $b=1$. We specify the minimum magnitude of the aftershocks to be $m_{\min}=0$. We have shown [4] that the choice of the minimum magnitude does not influence the statistics of the larger aftershocks. A minimum must be prescribed to keep the number of aftershocks finite. We take the magnitude difference $\Delta m^*=1.2$. This is the typical value found for actual aftershock sequences [12]. For the time dependence we take $p=1.25$ and $\tau_0=1$ as discussed above. The results of a typical simulation are given in Fig. 1. The magnitudes of the aftershocks m are given as a function of the nondimensional time $T=t/\tau_0$ after the mainshock. Six generations of aftershocks were generated in the sequence. Typical values of τ_0 for actual sequences are near $\tau_0=10^{-3}$ days [14]. Thus the simulation represents the first 100 days of aftershock activity. This simulation is similar to observed sequences [14].

We next consider more unstable sequences of events. We take the magnitude m_p of the original event to be $m_p=1$, $b=1$, and specify the minimum magnitude to be $m_{\min}=0$. For our example we take $\Delta m^*=-0.2$ so that the largest expected daughter event is 0.2 magnitude units larger than the parent event. For the time dependence we again take $p=1.25$ and $\tau_0=1$. The results of our simulation are given in Fig. 2. The magnitudes of the events are given as a function of the time T after the initial $m_p=1$ parent event. This sequence was terminated at $T=7$ at which time $N=19\,000$ events had occurred including 26 generations.

As we have shown in Figs. 1 and 2, we find sequences of events that die out and that grow exponentially. We next determine where the transition to likely blowup (exponential growth) occurs. For a specified value of Δm^*, we use simulations to determine the extinction probability q, and the probability of blowup $1-q$. We have run 100 simulations for various values of Δm^*. In Fig. 3 we plot the fraction of

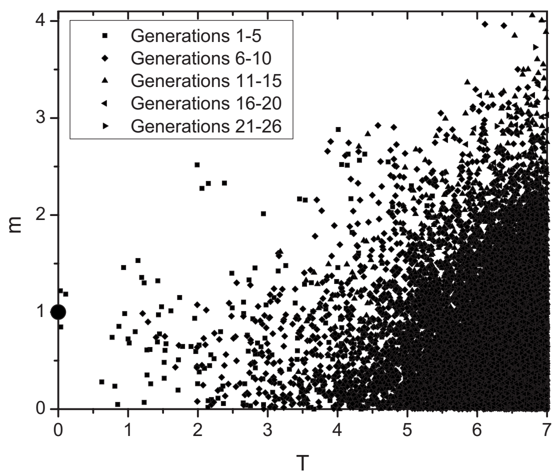

FIG. 2. Simulation of an unstable sequence with $\Delta m^*=-0.2$ and a $m_p=1$ initial parent event. Magnitudes m of the subsequent events are given as a function of the nondimensional time $T=t/\tau_0$. The sequence was terminated at $T=7$ at which time $N=19\,000$ events had occurred in 26 generations.

exponentially growing events $1-q$ versus Δm^*. The transition from likely die out to likely exponential growth occurs in the range $\Delta m^*=0.6$ to 0.2. For $\Delta m^*=0.36$ one-half of the simulations die out and one half grow exponentially; that is, $q=1-q=0.5$. We write $(\Delta m^*)_b=0.36$ for this transitional value.

We next determine the time dependent behavior for several exponentially growing sequences. In Figs. 4 we give the cumulative number of events N as a function of time. In Fig. 4(a) we give a sequence for $\Delta m^*=0.1$, in Fig. 4(b) we give a sequence for $\Delta m^*=-0.2$, and in Fig. 4(c) we give a sequence for $\Delta m^*=-0.4$. After initial transients each sequence settles into well defined exponential growth defined by

$$N = N_0 e^{\lambda T}. \tag{12}$$

For the three simulations illustrated in Fig. 4, we have $\lambda=0.379$ for $\Delta m^*=0.1$, $\lambda=1.25$ for $\Delta m^*=-0.2$, and $\lambda=2.02$ for $\Delta m^*=-0.4$. The growth rate increases systematically with decreasing values of Δm^*. In Fig. 5 we give the dependence of the exponential growth factor λ on Δm^*. We see

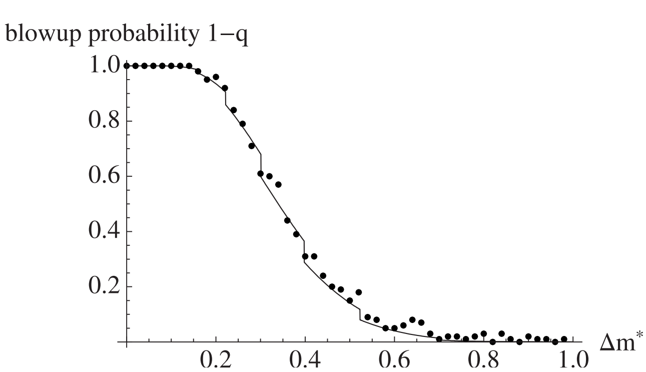

FIG. 3. Blowup probability $1-q$ as a function of Δm^*. The dots are obtained by simulation as the fraction $1-q$ of event simula tions that are unstable. The line is obtained by the calculations in Sec. V.

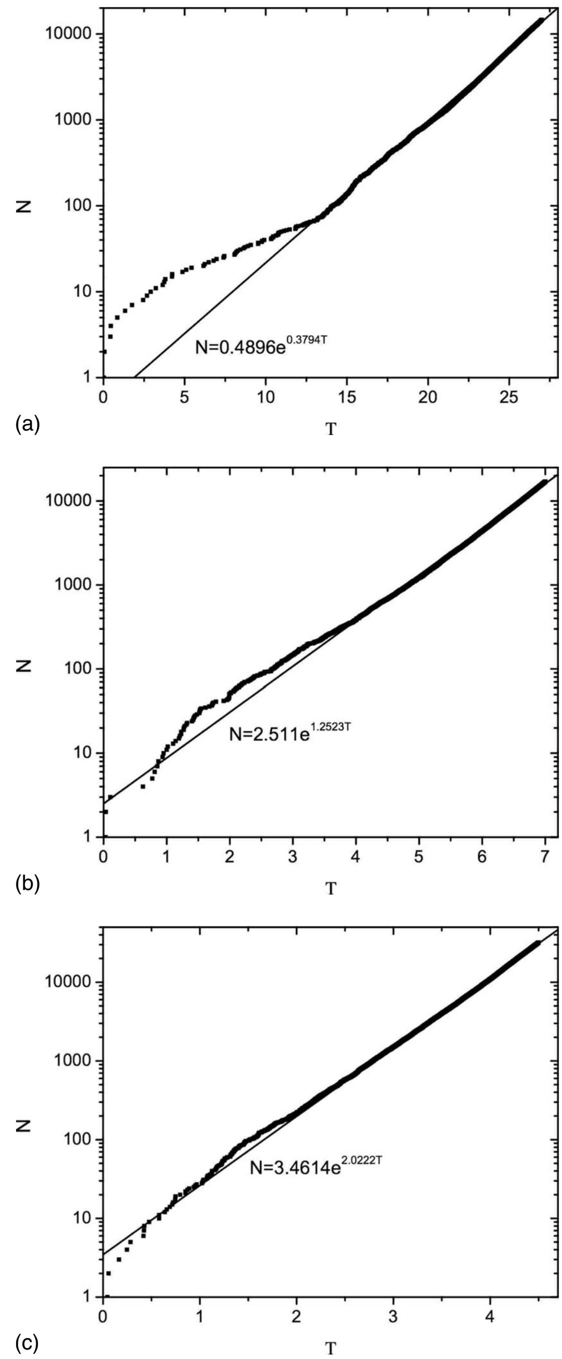

FIG. 4. Number of events as a function of nondimensional time $T=t/\tau_0$. The straight-line correlations are with the exponential growth relation (12). (a) $\Delta m^*=0.1$, (b) $\Delta m^*=-0.2$, and (c) $\Delta m^*=-0.4$.

that our results correlate quite well with the power law relation

$$\lambda = 3(-\Delta m^* + (\Delta m^*)_b)^{1.5}. \tag{13}$$

It must be emphasized that even with $\Delta m^* < (\Delta m^*)_b$, a significant fraction of the simulations diverge. We will next study this behavior using an analytical approach.

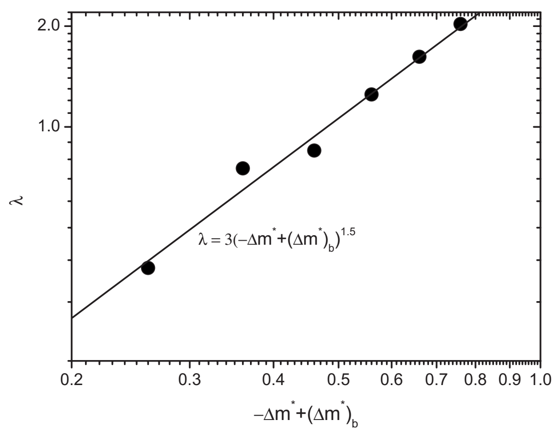

FIG. 5. Dependence of the exponential growth factor λ on the difference between Δm^* and its transitional value $(\Delta m^*)_b=0.36$.

V. ANALYTIC RESULTS FOR PROBABILITY OF BLOWUP

A major emphasis of this paper has been the transition from extinction to blowup. This has been quantified by the extinction probability q which is the fraction of simulations in which the number of events is finite. The blowup probability is then $1-q$ as shown in Fig. 3. In this section we calculate the extinction probability q analytically.

We consider one earthquake and regard the variation in its size as a random variable affecting the number of its daughter aftershocks. Then there is a probability distribution for the number of daughter aftershocks. Since each aftershock produced at any generation has a size independent of the other aftershocks at that generation, this produces a Galton-Watson branching process in the number of aftershocks at each generation that we now determine. The blowup and extinction probabilities of the branching process are then computed. If the branching process extinguishes, this implies a finite number of aftershocks. This branching process contains no information about the timing of the aftershocks and hence no information about the growth or decay rate of the aftershocks. However, the possibility of a finite number of aftershocks is a significant property of the model, and in practice controls the behavior of the model when a finite number of aftershocks is highly probable.

Suppose we consider one particular aftershock that occurs at any generation after the first generation earthquake. The size M of this aftershock is distributed according to Eq. (5) or, equivalently, the shifted exponential probability distribution function

$$f_M(m) = (b \ln 10)10^{-b(m-m_{\min})}, \quad m \geq m_{\min}. \tag{14}$$

Then, according to Eq. (4), the aftershock of size M will produce the number N of aftershocks in the next generation, where

$$N(M) = \lfloor 10^{b(M-\Delta m^*-m_{\min})} \rfloor \tag{15}$$

and $\lfloor x \rfloor$ is integer part of x.

The probability that the aftershock has n aftershocks in the next generation is

$$\begin{aligned} p_n &= P[N=n] = P[n \leq 10^{b(M-\Delta m^*-m_{\min})} < n+1] \\ &= P[b^{-1}\log_{10} n + \Delta m^* + m_{\min} \leq M < b^{-1}\log_{10}(n+1) + \Delta m^* + m_{\min}] \\ &= \int_{b^{-1}\log_{10} n+\Delta m^*+m_{\min}}^{b^{-1}\log_{10}(n+1)+\Delta m^*+m_{\min}} f_M(m)dm \\ &= b \ln 10 \int_{\max\{b^{-1}\log_{10} n+\Delta m^*,0\}}^{\max\{b^{-1}\log_{10}(n+1)+\Delta m^*,0\}} 10^{-bx}dx \\ &= \begin{cases} \max\{1-10^{-b\Delta m^*},0\}, & n=0, \\ [\max\{n10^{b\Delta m^*},1\}]^{-1} - [\max\{(n+1)10^{b\Delta m^*},1\}]^{-1}, & n \geq 1. \end{cases} \end{aligned} \tag{16}$$

For $\Delta m^* \geq 0$, Eq. (16) reduces to

$$p_n = \begin{cases} 1-10^{-b\Delta m^*}, & n=0, \\ \dfrac{10^{-b\Delta m^*}}{n(n+1)}, & n \geq 1. \end{cases} \tag{17}$$

The probability that the aftershock gives rise to no aftershocks in the next generation is p_0. When $\Delta m^* \geq 0$, Eq. (17) shows the exponential dependence of p_0 on Δm^*. Moreover, when $\Delta m^* \geq 0$, the probability of the aftershock having one or more children is $1-p_0=10^{-b\Delta m^*}$. When $\Delta m^* < 0$, Eq. (16) shows that $p_0=0$ and hence that aftershocks always have children. In this case, the sequence of aftershocks can never extinguish and blowup is certain. The generating function $f(s)$ for the branching process after the first generation is [2]

$$f(s) = \sum_{k=0}^{\infty} p_k s^k. \tag{18}$$

We now consider the first generation of aftershocks of a parent earthquake of given (deterministic) size M_1. Then for the first generation there are N_1 aftershocks, where

$$N_1 = N(M_1) = \lfloor 10^{b(M_1-\Delta m^*-m_{\min})} \rfloor. \tag{19}$$

The generating function $f_1(s)$ for the first generation of the branching process is

$$f_1(s) = s^{N_1} = s^{\lfloor 10^{b(M_1-\Delta m^*-m_{\min})} \rfloor}. \tag{20}$$

Notice that f_1 has jumps as functions of its arguments or parameters because the integer part function $\lfloor\cdot\rfloor$ has jumps.

Note that Eq. (16) or (17) shows that the probability of an aftershock having n daughters $p_n \sim n^{-2}$ for large n. Therefore each aftershock produces, in the next generation, a mean number of daughter aftershocks

$$\bar{N} = \sum_{n=0}^{\infty} n p_n = \infty. \tag{21}$$

The result (21) is independent of the values of Δm^* and b and the branching process is always supercritical.

We now state how to compute the extinction probability q [2]. First we compute the extinction probability q_* that assumes that there is one initial earthquake. q_* is the smallest non-negative root of

$$s = f(s). \tag{22}$$

[To solve Eq. (22), one can either use computer algebra to approximate the generating function f with the first 500 terms of the series (18) and then solve the resulting polynomial (22) numerically, or one can solve Eq. (22) iteratively by $q_* \approx f^{(k)}(0)$, where the exponent (k) indicates a suitable number of functional compositions of f such as $k=50$. These two methods yield the same results.] For our branching process we always have $p_0 \leq q_* < 1$. There is always a root $s=1$ of Eq. (22) larger than q_* and the mean offspring $\bar{N} = f'(1)$ is infinite. Nevertheless, q_* can be close to 1.

Then

$$q = f_1(q_*) = q_*^{N_1} = q_*^{\lfloor 10^{b(M_1-\Delta m^*-m_{\min})} \rfloor}. \tag{23}$$

q is the extinction probability for N_1 initial earthquakes and this extinction occurs precisely when each of the N_1 initial earthquakes produces a sequence of events that extinguishes. Since q_* is the extinction probability assuming one initial earthquake, and the branching process assumes that each of the N_1 initial earthquakes independently produces daughters, the probability q of all N_1 sequences extinguishing is given by Eq. (23). Equation (23) shows that the extinction probability q has a strong dependence on M_1 and Δm^*. Small M_1 or large Δm^* tend to make the probability of extinction high and the probability of exponential blowup small.

To show how the theory matches simulation results of the previous section, we take the case $b=1$, $m_{\min}=0$, $M_1=1$. For this case,

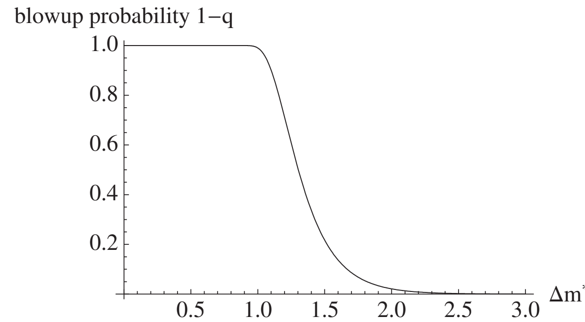

FIG. 6. Blowup probability $1-q$ as a function of Δm^* obtained by the calculations in Sec. V with $b=1$, $m_{\min}=0$, $M_1=5$.

$$N_1 = \lfloor 10^{1-\Delta m^*} \rfloor \quad \text{and} \quad p_0 = \max\{1-10^{-\Delta m^*}, 0\}. \tag{24}$$

In Fig. 3 we compare the results of this analysis with the simulation results. We see that the agreement is excellent. For $\Delta m^* > 1$, Eq. (24) shows that there are no aftershocks at all and the probability of blowup $1-q$ is zero. For $\Delta m^* = 0.9$, there can be aftershocks, and there is a possibility of blowup, but the probability of blowup is very small, with $1-q=0.0009$. For $\Delta m^* < 0$, since the probability of extinction in the next generation $p_0=0$, extinction is impossible and blowup is certain with $1-q=1$.

We also consider the blowup probability for the conditions used to generate the stable sequence of events illustrated in Fig. 1. Taking $M_1=5$, $b=1$, and $m_{\min}=0$, the blowup probability $1-q$ is given as a function of Δm^* in Fig. 6. For the example given in Fig. 1, $\Delta m^*=1.2$, and from Fig. 6 we see that the blowup probability $1-q=0.7$. However, 10 000 simulations for these conditions gave no examples of blowup for simulation times up to $T=10^6$.

VI. BASS, ETAS, AND BLOWUP

We first show how the BASS model can be derived from the ETAS model. In the ETAS model the total number of daughter earthquakes N_{dt} is given by the productivity relation [1]

$$N_{dt} = k 10^{\alpha(m_p - m_{\min})}. \tag{25}$$

In the BASS model Eq. (25) is replaced by Eq. (4). We see that Eq. (25) is identical to Eq. (4) if

$$\alpha = b \quad \text{and} \quad k = 10^{-b\Delta m^*}. \tag{26}$$

The ETAS model uses the GR scaling (3) and Omori's law (6) exactly as in the BASS model. Taking $\alpha \neq b$ in the ETAS model eliminates self-similarity [15].

A major problem is that the ETAS model introduces two constants k and α in Eq. (25) that are not constrained by observations. In the BASS model these constants are replaced with b and Δm^* as shown in Eq. (26). Both b and Δm^* are constrained by data [4]. Also there are two widely accepted observations that are not consistent with the ETAS model. The first is the applicability of Bath's law and its modified form. In the ETAS model there is an exponential dependence of Δm^* on main shock magnitude that is not consistent with observations. The second observation concerns foreshocks. The self-similar behavior of the BASS model predicts that the probability of an earthquake having a foreshock is independent of the main shock magnitude, whereas the ETAS model predicts an exponential dependence. The exponential dependence is not consistent with observations. The BASS model also predicts the modified form of the GR statistics whereas the ETAS model does not.

When Δm^* has the value 1.2 typical for earthquake aftershocks, there is a finite probability of blowup as illustrated in Fig. 6. Blowup cannot occur in an aftershock sequence because the available elastic energy is finite. This probability of blowup has led advocates of the ETAS model [3] to reject its self-similar limit, the BASS model. However, extensive studies of the BASS model [4] have shown that it is relatively easy to suppress the blowup of the BASS model that is studied in this paper. This can be done either by finite-time cutoffs or limits on the magnitude of a daughter earthquake relative to the size of the parent earthquake.

VII. DISCUSSION

In this paper we have applied a self-similar branching model that, although exponentially unstable on average, has a significant chance of extinction. The rationale for the model comes from studies of aftershock sequences. These sequences relax the stresses introduced by the mainshock in an earthquake sequence. In our self-similar branching model, the probability of extinction of the process is directly related to the control parameter Δm^* and we have systematically studied the transition from almost certain extinction to almost certain blowup.

When Δm^* has the value 1.2 typical for earthquake aftershocks, and there is an initial earthquake of modest size, the probability of the process blowing up exponentially is small and the probability of the process dying out is high. In terms of the earthquake analogy the sequence of aftershocks relaxes the high stresses imposed by the original mainshock. For a larger initial earthquake, the process is in a transitional region where dying out and ultimately blowing up exponentially are both likely possibilities. However, the exponential blowup may not be evident in finite-time simulations.

When Δm^* is small or negative in our branching model the probability of the process dying out is small and it is likely that the number of events increases exponentially with time. The magnitude of the largest event increases linearly with time. This instability is a multiplicative growth process. Large events trigger even larger events.

The self-similar branching process considered in this paper is analogous to other self-similar branching processes, both in simulations and in nature. One example is river networks [16]. Many branching examples in biology behave in direct analogy to river networks. Examples are pulmonary systems, cardiovascular systems, and the vein structure of leaves [17]. A simulation that gives a similar behavior is diffusion limited aggregation [18]. All these examples including our self-similar branching model satisfy Tokunaga fractal side-branching statistics [19].

It is of interest to contrast the multiplicative exponential growth phenomena discussed above with critical point phenomena. This is most easily done by considering a specific example, site percolation. A percolating cluster is a classical critical point with scaling exponents [20]. One way to approach the critical point is to utilize a forest-fire model without fires. Trees are randomly planted on sites until a continuous path of trees cross the grid; this is the critical point. It has been shown [21] that the growth of tree clusters on the approach to criticality is dominated by cluster coalescence. Planted trees bridge gaps between tree clusters to create larger clusters. It has been shown analytically [22] that this additive coalescence process directly gives the power-law scaling associated with the approach to a critical point. Other classical critical point problem such as the Ising model, magnetization, and phase changes are directly associated with cluster coalescence. Cluster coalescence is fundamentally an additive process. Large clusters evolve from the coalescence of smaller clusters. This is in direct contrast to the multiplicative branching processes considered above. In our self-similar branch model large events are created instantaneously and do not evolve from the coalescence of smaller events.

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[1] Y. Ogata, J. Am. Stat. Assoc. **83**, 9 (1988); A. Helmstetter and D. Sornette, Phys. Rev. E **66**, 061104 (2002); J. Geophys. Res. **108**, 2457 (2003); Y. Ogata and H. C. Zhuang, Tectonophysics **413**, 13 (2006).

[2] T. E. Harris, *Theory of Branching Processes* (Dover, New York, 1989); K. B. Athreya and P. E. Ney, *Branching Processes* (Dover, New York, 2004).

[3] A. Helmstetter and D. Sornette, J. Geophys. Res. **107**, 2237 (2002); D. Sornette and M. J. Werner, *ibid.* **110**, B08304 (2005).

[4] D. L. Turcotte, J. R. Holliday, and J. B. Rundle, Geophys. Res. Lett. **34**, L12303 (2007); J. R. Holliday, D. L. Turcotte, and J. B. Rundle, Physica A **387**, 933 (2008); Pure Appl. Geophys. **165**, 1003 (2008).

[5] R. Shcherbakov, D. L. Turcotte, and J. B. Rundle, Pure Appl. Geophys. **162**, 1051 (2005).

[6] B. Gutenberg and C. F. Richter, *Seismicity of the Earth and Associated Phenomenon*, 2nd ed. (Princeton University Press, Princeton, 1954).

[7] D. L. Turcotte, *Fractals and Chaos in Geology and Geophys-*

ics, 2nd ed. (Cambridge University Press, Cambridge, 1997).
[8] T. Utsu, Geophys. Mag. **30**, 521 (1961).
[9] R. Shcherbakov, D. L. Turcotte, and J. B. Rundle, Geophys. Res. Lett. **31**, L11613 (2004).
[10] M. Bath, Tectonophysics **2**, 483 (1965).
[11] Y. Ogata, T. Utsu, and K. Katsura, Geophys. J. Int. **121**, 233 (1995).
[12] R. Shcherbakov and D. L. Turcotte, Bull. Seismol. Soc. Am. **94**, 1968 (2004).
[13] H. Kanamori and D. L. Anderson, Bull. Seismol. Soc. Am. **65**, 1073 (1975).
[14] K. Z. Nanjo, B. Enescu, R. Shcherbakov, D. L. Turcotte, T. Iwata, and Y. Ogata, J. Geophys. Res. **112**, B08309 (2007).
[15] K. R. Felzer, R. E. Abercrombie, and G. Ekstrom, Bull. Seismol. Soc. Am. **94**, 88 (2004).
[16] J. D. Pelletier, J. Geophys. Res. **104**, 7359 (1999).
[17] D. L. Turcotte, J. D. Pelletier, and W. I. Newman, J. Theor. Biol. **193**, 577 (1998).
[18] P. Ossadnik, Phys. Rev. A **45**, 1058 (1992).
[19] W. I. Newman, D. L. Turcotte, and A. M. Gabrielov, Fractals **5**, 603 (1997).
[20] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, New York, 1992), p. 181.
[21] D. L. Turcotte, B. D. Malamud, G. Morein, and W. I. Newman, Physica A **268**, 629 (1999); D. L. Turcotte, Rep. Prog. Phys. **62**, 1377 (1999).
[22] A. Gabrielov, W. I. Newman, and D. L. Turcotte, Phys. Rev. E **60**, 5293 (1999).